

# Broadcasting Correlated Gaussians

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## Abstract

We consider the transmission of a memoryless bivariate Gaussian source over an average-power-constrained one-to-two Gaussian broadcast channel. The transmitter observes the source and describes it to the two receivers by means of an average-power-constrained signal. Each receiver observes the transmitted signal corrupted by a different additive white Gaussian noise and wishes to estimate the source component intended for it. That is, Receiver 1 wishes to estimate the first source component and Receiver 2 wishes to estimate the second source component. Our interest is in the pairs of expected squared-error distortions that are simultaneously achievable at the two receivers.

We prove that an uncoded transmission scheme that sends a linear combination of the source components achieves the optimal power-versus-distortion trade-off whenever the signal-to-noise ratio is below a certain threshold. The threshold is a function of the source correlation and the distortion at the receiver with the weaker noise.

## 1 Introduction

We consider the transmission of a memoryless bivariate Gaussian source over an average-power-constrained one-to-two Gaussian broadcast channel. The transmitter observes the source and describes it to the two receivers by means of an average-power-constrained signal. Each receiver observes the transmitted signal corrupted by a different additive white Gaussian noise and wishes to estimate the source component intended for it. That is, Receiver 1 wishes to estimate the first source component and Receiver 2 wishes to estimate the second source component. Our interest is in the pairs of expected squared-error distortions that are simultaneously achievable at the two receivers.

We prove that an uncoded transmission scheme that sends a linear combination of the source components achieves the optimal power-versus-distortion trade-off whenever the signal-to-noise ratio is below a certain threshold. The threshold is a function of the source correlation and the distortion at the receiver with the weaker noise.

This result is reminiscent of the results in [1, 2] about the optimality of uncoded transmission of a bivariate Gaussian source over a Gaussian multiple-access channel, without and with feedback. There too, uncoded transmission is optimal below a certain SNR-threshold. This work is also related to the classical result of Gobblick [3], who showed that for the transmission of a memoryless Gaussian source over the additive

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white Gaussian noise channel, the minimal expected squared-error distortion is achieved by an uncoded transmission scheme. It is also related to the work of Gastpar [4] who showed for some combined source-channel coding analog of the quadratic Gaussian CEO problem that the minimal expected squared-error distortion is achieved by an uncoded transmission scheme.

## 2 Problem Statement

Our setup is illustrated in Figure 1. It consists of a memoryless bivariate Gaussian

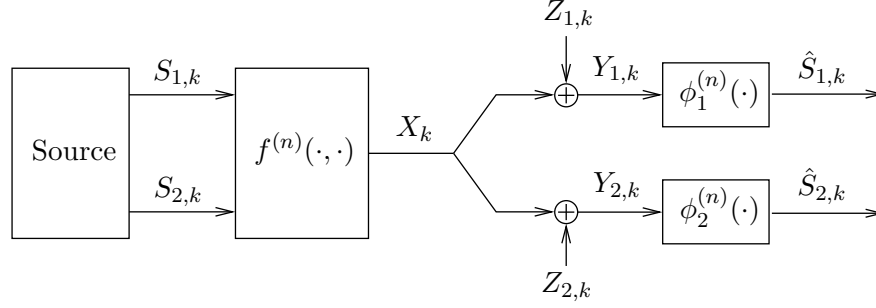


Figure 1: Two-user Gaussian broadcast channel with bivariate source.

source and a one-to-two Gaussian broadcast channel. The memoryless source emits at each time  $k \in \mathbb{Z}$  a bivariate Gaussian  $(S_{1,k}, S_{2,k})$  of zero mean and covariance matrix<sup>1</sup>

$$\mathbf{K}_{SS} = \sigma^2 \begin{pmatrix} 1 & \rho \\ \rho & 1 \end{pmatrix}, \quad \text{where } \rho \in [0, 1). \quad (1)$$

The source is to be transmitted over a memoryless Gaussian broadcast channel with time- $k$  input  $x_k \in \mathbb{R}$ , which is subjected to an expected average power constraint

$$\frac{1}{n} \sum_{k=1}^n \mathbb{E}[X_k^2] \leq P, \quad (2)$$

for some given  $P > 0$ . The time- $k$  output  $Y_{i,k}$  at Receiver  $i$  is given by

$$Y_{i,k} = x_k + Z_{i,k} \quad i \in \{1, 2\},$$

where  $Z_{i,k}$  is the time- $k$  additive noise term on the channel to Receiver  $i$ . For each  $i \in \{1, 2\}$  the sequence  $\{Z_{i,k}\}_{k=1}^\infty$  is independent identically distributed (IID)  $\mathcal{N}(0, N_i)$  and independent of the source sequence  $\{(S_{1,k}, S_{2,k})\}$ , where  $\mathcal{N}(\mu, \nu^2)$  denotes the mean- $\mu$  variance- $\nu^2$  Gaussian distribution and where we assume that<sup>2</sup>

$$N_1 < N_2. \quad (3)$$

For the transmission we consider block encoding schemes where, for blocklength  $n$ , the transmitted sequence  $\mathbf{X} = (X_1, X_2, \dots, X_n)$  is given by

$$\mathbf{X} = f^{(n)}(\mathbf{S}_1, \mathbf{S}_2), \quad (4)$$

<sup>1</sup>The restrictions made on  $\mathbf{K}_{SS}$ , i.e., that  $\rho \in [0, 1)$  and that  $\text{Var}(S_{1,k}) = \text{Var}(S_{2,k}) = \sigma^2$  will be justified in Remark 2.2, once the problem has been stated completely.

<sup>2</sup>The case  $N_1 = N_2$  is equivalent to the problem of sending a bivariate Gaussian on a single-user Gaussian channel [1].

for some encoding function  $f^{(n)}: \mathbb{R}^n \times \mathbb{R}^n \rightarrow \mathbb{R}^n$ , and where we use boldface characters to denote  $n$ -tuples, e.g.  $\mathbf{S}_1 = (S_{1,1}, S_{1,2}, \dots, S_{1,n})$ . Receiver  $i$ 's estimate  $\hat{\mathbf{S}}_i$  of the source sequence  $\mathbf{S}_i$  intended for it, is a function  $\phi_i^{(n)}: \mathbb{R}^n \rightarrow \mathbb{R}^n$  of its observation  $\mathbf{Y}_i$ ,

$$\hat{\mathbf{S}}_i = \phi_i^{(n)}(\mathbf{Y}_i) \quad i \in \{1, 2\}. \quad (5)$$

The quality of the estimate  $\hat{\mathbf{S}}_i$  with respect to the original source sequence  $\mathbf{S}_i$  is measured in expected squared-error distortion averaged over the blocklength  $n$ . We denote this distortion by  $\delta_i^{(n)}$ , i.e.

$$\delta_i^{(n)} \triangleq \frac{1}{n} \sum_{k=1}^n \mathbb{E} \left[ (S_{i,k} - \hat{S}_{i,k})^2 \right] \quad i \in \{1, 2\}. \quad (6)$$

Our interest is in the set of distortion pairs that can be achieved simultaneously at the two receivers as the blocklength  $n$  tends to infinity. This notion of achievability is described more precisely in the following definition.

**Definition 2.1** (Achievability). Given  $\sigma^2 > 0$ ,  $\rho \in [0, 1]$ ,  $P > 0$  and  $0 < N_1 \leq N_2$ , we say that the tuple  $(D_1, D_2, \sigma^2, \rho, P, N_1, N_2)$  is *achievable* (or in short, that the pair  $(D_1, D_2)$  is achievable) if there exist a sequence of encoding functions  $\{f^{(n)}\}$  as in (4) satisfying the average power constraint (2) and sequences of reconstruction functions  $\{\phi_1^{(n)}\}$ ,  $\{\phi_2^{(n)}\}$  as in (5) with resulting average distortions  $\delta_1^{(n)}$ ,  $\delta_2^{(n)}$  as in (6) that fulfill

$$\overline{\lim}_{n \rightarrow \infty} \delta_i^{(n)} \leq D_i \quad i \in \{1, 2\},$$

whenever

$$\mathbf{Y}_i = f^{(n)}(\mathbf{S}_1, \mathbf{S}_2) + \mathbf{Z}_i \quad i \in \{1, 2\}, \quad (7)$$

for  $\{(S_{1,k}, S_{2,k})\}$  an IID sequence of zero-mean bivariate Gaussians with covariance matrix as in (1) and  $\{Z_{i,k}\}_{k=1}^\infty$  IID zero-mean Gaussians of variance  $N_i$ ,  $i \in \{1, 2\}$ .

Based on Definition 2.1, we next define the set of all achievable distortion pairs.

**Definition 2.2** ( $\mathcal{D}(\sigma^2, \rho, P, N_1, N_2)$ ). For any  $\sigma^2$ ,  $\rho$ ,  $P$ ,  $N_1$ , and  $N_2$  as in Definition 2.1 we define  $\mathcal{D}(\sigma^2, \rho, P, N_1, N_2)$  (or just  $\mathcal{D}$ ) as the region of all pairs  $(D_1, D_2)$  for which  $(D_1, D_2, \sigma^2, \rho, P, N_1, N_2)$  is achievable, i.e.

$$\mathcal{D}(\sigma^2, \rho, P, N_1, N_2) = \{(D_1, D_2) : (D_1, D_2, \sigma^2, \rho, P, N_1, N_2) \text{ is achievable}\}.$$

**Remark 2.1.** The region  $\mathcal{D}$  is closed and convex.

*Proof.* See Appendix A.1. □

**Remark 2.2.** In the description of the source law in (1), we have excluded the case where  $\rho = 1$ . We have done so because for this case the optimality of uncoded transmission follows immediately for all SNRs from the corresponding result for the single user scenario in [3]. Moreover, we have also assumed that the source components are of equal variance and that their correlation coefficient  $\rho$  is nonnegative. We now show that these two assumptions incur no loss in generality.

- i) We can limit ourselves to nonnegative correlation coefficients  $\rho$  because the distortion region  $\mathcal{D}$  depends on the correlation coefficient only via its absolute value  $|\rho|$ . That is, the tuple  $(D_1, D_2, \sigma^2, \rho, P, N_1, N_2)$  is achievable if, and only if, the tuple  $(D_1, D_2, \sigma^2, -\rho, P, N_1, N_2)$  is achievable. To see this, note that if  $\{f^{(n)}, \phi_1^{(n)}, \phi_2^{(n)}\}$  achieves the distortion  $(D_1, D_2)$  for the source of correlation coefficient  $\rho$ , then  $\{\tilde{f}_1^{(n)}, \tilde{\phi}_1^{(n)}, \phi_2^{(n)}\}$ , where

$$\tilde{f}_1^{(n)}(\mathbf{S}_1, \mathbf{S}_2) = f^{(n)}(-\mathbf{S}_1, \mathbf{S}_2) \quad \text{and} \quad \tilde{\phi}_1^{(n)}(\mathbf{Y}) = -\phi_1^{(n)}(\mathbf{Y})$$

achieves  $(D_1, D_2)$  for the source with correlation coefficient  $-\rho$ .

- ii) The restriction to source components of equal variances incurs no loss of generality because the distortion region scales linearly with the variance of the source components. To see this, consider the more general case where the two source components are not necessarily of equal variances, i.e., where  $\text{Var}(S_{1,k}) = \sigma_1^2$  and  $\text{Var}(S_{2,k}) = \sigma_2^2$  for some  $\sigma_1^2, \sigma_2^2 > 0$  and for all  $k \in \mathbb{Z}$ . Accordingly, define a tuple  $(D_1, D_2, \sigma_1^2, \sigma_2^2, \rho, P, N_1, N_2)$  to be achievable, similarly as in Definition 2.1. The proof now follows from showing that the tuple  $(D_1, D_2, \sigma_1^2, \sigma_2^2, \rho, P, N_1, N_2)$  is achievable if, and only if, for every  $\alpha_1, \alpha_2 \in \mathbb{R}^+$ , the tuple  $(\alpha_1 D_1, \alpha_2 D_2, \alpha_1 \sigma_1^2, \alpha_2 \sigma_2^2, \rho, P, N_1, N_2)$  is achievable. This can be seen as follows. If  $\{f^{(n)}, \phi_1^{(n)}, \phi_2^{(n)}\}$  achieves the tuple  $(D_1, D_2, \sigma_1^2, \sigma_2^2, \rho, P, N_1, N_2)$ , then  $\{\tilde{f}^{(n)}, \tilde{\phi}_1^{(n)}, \tilde{\phi}_2^{(n)}\}$  where

$$\tilde{f}^{(n)}(\mathbf{S}_1, \mathbf{S}_2) = f^{(n)}\left(\frac{\mathbf{S}_1}{\sqrt{\alpha_1}}, \frac{\mathbf{S}_2}{\sqrt{\alpha_2}}\right),$$

and where

$$\tilde{\phi}_i^{(n)}(\mathbf{Y}) = \sqrt{\alpha_i} \cdot \phi_i^{(n)}(\mathbf{Y}), \quad i \in \{1, 2\},$$

achieves the tuple  $(\alpha_1 D_1, \alpha_2 D_2, \alpha_1 \sigma_1^2, \alpha_2 \sigma_2^2, \rho, P, N_1, N_2)$ . And by an analogous argument it follows that if  $(\alpha_1 D_1, \alpha_2 D_2, \alpha_1 \sigma_1^2, \alpha_2 \sigma_2^2, \rho, P, N_1, N_2)$  is achievable, then also  $(D_1, D_2, \sigma_1^2, \sigma_2^2, \rho, P, N_1, N_2)$  is achievable.

We state one more property of the region  $\mathcal{D}$ . To this end, we need the following two definitions.

**Definition 2.3** ( $D_{i,\min}$ ). We say that  $D_1$  is achievable if there exists some  $D_2$  such that  $(D_1, D_2) \in \mathcal{D}$ . The smallest achievable  $D_1$  is denoted by  $D_{1,\min}$ . The achievability of  $D_2$  and the distortion  $D_{2,\min}$  are analogously defined.

By the classical single-user result [5, Theorem 9.6.3, p. 473]

$$D_{i,\min} \triangleq \sigma^2 \frac{N_i}{N_i + P} \quad i \in \{1, 2\}.$$

**Definition 2.4** ( $D_1^*(D_2)$  and  $D_2^*(D_1)$ ). For every achievable  $D_2$ , we define  $D_1^*(D_2)$  as the smallest  $D_1'$  such that  $(D_1', D_2)$  is achievable, i.e.,

$$D_1^*(D_2) \triangleq \min \{D_1' : (D_1', D_2) \in \mathcal{D}\}.$$

Similarly,

$$D_2^*(D_1) \triangleq \min \{D_2' : (D_1, D_2') \in \mathcal{D}\}.$$

In general, we have no closed-form expression for  $D_1^*(\cdot)$  and  $D_2^*(\cdot)$ . However, in the following two special cases we do:

**Proposition 2.1.** *The distortion  $D_1^*(D_{2,\min})$  is given by*

$$D_1^*(D_{2,\min}) = \sigma^2 \frac{N_1 + P(1 - \rho^2)}{N_1 + P}. \quad (8)$$

*The distortion pair  $(D_1^*(D_{2,\min}), D_{2,\min})$  is achieved by setting  $X_k = \sqrt{P/\sigma^2} S_{2,k}$ .*

*Proof.* See Appendix A.2. □

**Proposition 2.2.** *The distortion  $D_2^*(D_{1,\min})$  is given by*

$$D_2^*(D_{1,\min}) = \sigma^2 \frac{N_2 + P(1 - \rho^2)}{N_2 + P}. \quad (9)$$

*The distortion pair  $(D_{1,\min}, D_2^*(D_{1,\min}))$  is achieved by setting  $X_k = \sqrt{P/\sigma^2} S_{1,k}$ .*

*Proof.* The value of  $D_2^*(D_{1,\min})$  follows from Theorem 3.1 ahead as follows: For  $D_1 = D_{1,\min}$  it can be verified that condition (13) of Theorem 3.1 is satisfied for all  $P/N_1$ . Hence, the pair  $(D_{1,\min}, D_2^*(D_{1,\min}))$  is always achieved by the uncoded scheme with  $\alpha = 1$ ,  $\beta = 0$ , and so

$$D_2^*(D_{1,\min}) = \sigma^2 \frac{N_2 + P(1 - \rho^2)}{N_2 + P}. \quad \square$$

(This remark will not be used in the proof of Theorem 3.1.)

### 3 Main Result

Our main result states that, below a certain SNR-threshold, every pair  $(D_1, D_2) \in \mathcal{D}$  can be achieved by an uncoded scheme, where for every time-instant  $1 \leq k \leq n$ , the channel input is of the form

$$X_k^u(\alpha, \beta) = \sqrt{\frac{P}{\sigma^2(\alpha^2 + 2\alpha\beta\rho + \beta^2)}} (\alpha S_{1,k} + \beta S_{2,k}), \quad (10)$$

for some  $\alpha, \beta \in \mathbb{R}$ . The estimate  $\hat{S}_{i,k}^u$  of  $S_{i,k}$  (at Receiver  $i$ ),  $i \in \{1, 2\}$ , is the minimum mean squared-error estimate of  $S_{i,k}$  based on the scalar observation  $Y_{i,k}$ , i.e.,

$$\hat{S}_{i,k}^u = \mathbb{E}[S_{i,k}|Y_{i,k}], \quad i \in \{1, 2\}.$$

We denote the distortions resulting from this uncoded scheme by  $D_1^u$  and  $D_2^u$ . They are given by

$$D_1^u(\alpha, \beta) = \sigma^2 \frac{P^2\beta^2(1 - \rho^2) + PN_1(\alpha^2 + 2\alpha\beta\rho + \beta^2(2 - \rho^2)) + N_1^2(\alpha^2 + 2\alpha\beta\rho + \beta^2)}{(P + N_1)^2(\alpha^2 + 2\alpha\beta\rho + \beta^2)}, \quad (11)$$

$$D_2^u(\alpha, \beta) = \sigma^2 \frac{P^2\alpha^2(1 - \rho^2) + PN_2(\alpha^2(2 - \rho^2) + 2\alpha\beta\rho + \beta^2) + N_2^2(\alpha^2 + 2\alpha\beta\rho + \beta^2)}{(P + N_2)^2(\alpha^2 + 2\alpha\beta\rho + \beta^2)}. \quad (12)$$

**Remark 3.1.** In the reminder, we shall limit ourselves to transmission schemes with  $\alpha \in [0, 1]$  and  $\beta = 1 - \alpha$ . This incurs no loss in optimality, as we next show. For  $\rho \geq 0$ , an uncoded transmission scheme with the choice of  $(\alpha, \beta)$  such that  $\alpha\beta < 0$ , yields a distortion that is uniformly worse than the choice  $(|\alpha|, |\beta|)$ . Thus, without loss in optimality, we can restrict ourselves to  $\alpha, \beta \geq 0$ . It remains to notice that for  $\alpha, \beta \geq 0$ , the channel input  $X_k^u(\alpha, \beta)$  depends on  $\alpha, \beta$  only via the ratio  $\alpha/\beta$ .

Our main result can now be stated as follows.

**Theorem 3.1.** *For every  $(D_1, D_2) \in \mathcal{D}$  and*

$$\frac{P}{N_1} \leq \Gamma(D_1, \sigma^2, \rho), \quad (13)$$

*there exist  $\alpha^*, \beta^* \geq 0$  such that*

$$D_1^u(\alpha^*, \beta^*) \leq D_1 \quad \text{and} \quad D_2^u(\alpha^*, \beta^*) \leq D_2,$$

*where the threshold  $\Gamma$  is given by*

$$\Gamma(D_1, \sigma^2, \rho) = \begin{cases} \frac{\sigma^4(1-\rho^2)-2D_1\sigma^2(1-\rho^2)+D_1^2}{D_1(\sigma^2(1-\rho^2)-D_1)} & \text{if } 0 < D_1 < \sigma^2(1-\rho^2), \\ +\infty & \text{otherwise.} \end{cases}$$

*Proof.* See Appendix B. □

For  $0 < D_1 < \sigma^2(1-\rho^2)$  the threshold function satisfies  $\Gamma \geq 2\rho/(1-\rho)$  where equality is satisfied for  $D_1 = \sigma^2(1-\rho)$ . Thus a weaker, but simpler, form of Theorem 3.1 is

**Corollary 3.1.** *If*

$$\frac{P}{N_1} \leq \frac{2\rho}{1-\rho}, \quad (14)$$

*then any  $(D_1, D_2) \in \mathcal{D}$  is achievable by the uncoded scheme, i.e. for every  $(D_1, D_2) \in \mathcal{D}$  there exist some  $\alpha^*, \beta^* \geq 0$  such that*

$$D_1^u(\alpha^*, \beta^*) \leq D_1 \quad \text{and} \quad D_2^u(\alpha^*, \beta^*) \leq D_2.$$

## 4 Summary

We studied the transmission of a memoryless bivariate Gaussian source over an average-power-constrained one-to-two Gaussian broadcast channel. In this problem, the transmitter of the channel observes the source and describes it to the two receivers by means of an average-power-constrained signal. Each receiver observes the transmitted signal corrupted by a different additive white Gaussian noise and wishes to estimate one of the source components. That is, Receiver 1 wishes to estimate the first source component and Receiver 2 wishes to estimate the second source component. Our interest was in the pairs of expected squared-error distortions that are simultaneously achievable at the two receivers.

For this problem, we presented the optimality of an uncoded transmission scheme for all SNRs below a certain threshold (see Theorem 3.1). A weaker form of this result (see Corollary 3.1) is that if the SNR on the link with the weaker additive noise satisfies

$$\frac{P}{N_1} \leq \frac{2\rho}{1-\rho},$$

then every achievable distortion pair is achieved by the presented uncoded transmission scheme.

## A Proof of Remark 2.1 and Proposition 2.1

### A.1 Proof of Remark 2.1

The convexity of  $\mathcal{D}$  follows by a time-sharing argument. This technique is demonstrated in [6, Proof of Lemma 13.4.1, pp. 349].

We now prove that  $\mathcal{D}$  is closed. To this end, let  $\{\nu D_1\}_{\nu=1}^\infty$ ,  $\{\nu D_2\}_{\nu=1}^\infty$  be sequences satisfying  $(\nu D_1, \nu D_2) \in \mathcal{D}$ , for all  $\nu \in \mathbb{N}^+$ , and satisfying

$$\lim_{\nu \rightarrow \infty} \nu D_i = D_i \quad i \in \{1, 2\},$$

for some  $D_1, D_2 \in \mathbb{R}$ . To show that  $\mathcal{D}$  is closed we need to show that  $(D_1, D_2) \in \mathcal{D}$ . We construct a sequence of schemes achieving  $(D_1, D_2)$  as follows. Since  $(\nu D_1, \nu D_2) \in \mathcal{D}$ , it follows that there exists a monotonically increasing sequence of positive integers  $\{n_\nu\}_{\nu=1}^\infty$  such that for all  $n \geq n_\nu$  there exists a scheme  $(f_\nu^{(n)}, \phi_{1,\nu}^{(n)}, \phi_{2,\nu}^{(n)})$  satisfying

$$\begin{aligned} \frac{1}{n} \sum_{k=1}^n \mathbb{E} \left[ (S_{1,k} - \hat{S}_{1,k})^2 \right] &< \nu D_1 + \frac{1}{\nu}, \\ \frac{1}{n} \sum_{k=1}^n \mathbb{E} \left[ (S_{2,k} - \hat{S}_{2,k})^2 \right] &< \nu D_2 + \frac{1}{\nu}. \end{aligned}$$

Since  $n_\nu$  is increasing in  $\nu$ , we now choose our sequence of schemes to be  $\{f_\nu^{(n)}\}$ ,  $\{\phi_{1,\nu}^{(n)}\}$ ,  $\{\phi_{2,\nu}^{(n)}\}$  for all  $n \in [n_\nu, n_{\nu+1})$  and  $\nu \in \mathbb{N}^+$ . This sequence of schemes satisfies

$$\overline{\lim}_{n \rightarrow \infty} \frac{1}{n} \sum_{k=1}^n \mathbb{E} \left[ (S_{1,k} - \hat{S}_{1,k})^2 \right] \leq D_1, \quad (15)$$

$$\overline{\lim}_{n \rightarrow \infty} \frac{1}{n} \sum_{k=1}^n \mathbb{E} \left[ (S_{2,k} - \hat{S}_{2,k})^2 \right] \leq D_2, \quad (16)$$

so, by Definition 2.1, the pair  $(D_1, D_2)$  is achievable, i.e., in  $\mathcal{D}$ .  $\square$

### A.2 Proof of Proposition 2.1

To prove Proposition 2.1 we derive a lower bound on  $D_1^*(D_{2,\min})$  and then show that this lower bound is achieved by the uncoded scheme. To this end, let

$$\mathbf{W}_1 \triangleq \mathbf{S}_1 - \rho \mathbf{S}_2, \quad (17)$$

and note that  $\mathbf{W}_1$  is independent of  $\mathbf{S}_2$ . The key to the lower bound is that for any sequence of schemes achieving  $D_{2,\min}$ , the amount of information that  $\mathbf{Y}_1$  can contain about  $\mathbf{W}_1$  must vanish as  $n \rightarrow \infty$ . This will be stated more precisely later on.

Let  $\{f^{(n)}, \phi_1^{(n)}, \phi_2^{(n)}\}$  be some sequence of coding schemes achieving the distortion  $D_{2,\min}$  in the sense that

$$\overline{\lim}_{n \rightarrow \infty} \delta_2^{(n)} = D_{2,\min}, \quad (18)$$

where  $\delta_1^{(n)}$  and  $\delta_2^{(n)}$  are as in (6). Let  $\mathbf{X}$  be the channel input associated with this coding scheme, and let  $\mathbf{Y}_1$  be the resulting  $n$ -tuple received by Receiver 1.

We now lower bound  $\delta_1^{(n)}$  using the relation  $\mathbf{S}_1 = \mathbf{W}_1 + \rho \mathbf{S}_2$ . From this relation it follows that the optimal estimator, for  $1 \leq k \leq n$ , is

$$\begin{aligned} \mathbb{E}[S_{1,k}|\mathbf{Y}_1] &= \mathbb{E}[W_{1,k} + \rho S_{2,k}|\mathbf{Y}_1] \\ &= \mathbb{E}[W_{1,k}|\mathbf{Y}_1] + \rho \mathbb{E}[S_{2,k}|\mathbf{Y}_1]. \end{aligned}$$

Since  $\phi_1^{(n)}$  cannot outperform the optimal estimator,

$$\begin{aligned} \delta_1^{(n)} &\geq \frac{1}{n} \sum_{k=1}^n \mathbb{E}[(S_{1,k} - \mathbb{E}[S_{1,k}|\mathbf{Y}_1])^2] \\ &= \frac{1}{n} \sum_{k=1}^n \mathbb{E}[(W_{1,k} + \rho S_{2,k} - \mathbb{E}[W_{1,k}|\mathbf{Y}_1] - \rho \mathbb{E}[S_{2,k}|\mathbf{Y}_1])^2] \\ &= \rho^2 \frac{1}{n} \sum_{k=1}^n \mathbb{E}[(S_{2,k} - \mathbb{E}[S_{2,k}|\mathbf{Y}_1])^2] + \frac{1}{n} \sum_{k=1}^n \mathbb{E}[(W_{1,k} - \mathbb{E}[W_{1,k}|\mathbf{Y}_1])^2] \\ &\quad + 2\rho \frac{1}{n} \sum_{k=1}^n \mathbb{E}[(W_{1,k} - \mathbb{E}[W_{1,k}|\mathbf{Y}_1])(S_{2,k} - \mathbb{E}[S_{2,k}|\mathbf{Y}_1])]. \end{aligned} \quad (19)$$

We now lower bound the three terms on the RHS of (19). For the first term we have

$$\begin{aligned} \frac{1}{n} \sum_{k=1}^n \mathbb{E}[(S_{2,k} - \mathbb{E}[S_{2,k}|\mathbf{Y}_1])^2] &\geq \sigma^2 2^{-\frac{2}{n}I(\mathbf{S}_2;\mathbf{Y}_1)} \\ &\geq \sigma^2 2^{-\frac{2}{n}I(\mathbf{X};\mathbf{Y}_1)} \\ &\geq \sigma^2 \frac{N_1}{P + N_1}, \end{aligned} \quad (20)$$

where the first inequality follows by rate-distortion theory, the second inequality by the data processing inequality, and the third because the IID Gaussian input maximizes the mutual information.

To bound the second term in (19) we use the following lemma.

**Lemma A.1.** *For any sequence of schemes achieving  $D_{2,\min}$  in the sense of (18) and any  $\epsilon > 0$  there exists an integer  $n_\epsilon$  such that for all  $n \geq n_\epsilon$*

$$I(\mathbf{W}_1; \mathbf{Y}_1) \leq \frac{n}{2} \log_2 \left( \frac{\epsilon + N_1}{N_1} \right), \quad (21)$$

where  $\mathbf{W}_1$  is defined in (17) and  $\mathbf{Y}_1$  is the  $n$ -tuple received by Receiver 1 when this scheme is used.

*Proof.* See Appendix A.2.1. □

We now have

$$\begin{aligned} \frac{1}{n} \sum_{k=1}^n \mathbb{E}[(W_{1,k} - \mathbb{E}[W_{1,k}|\mathbf{Y}_1])^2] &\geq \sigma^2 (1 - \rho^2) 2^{-\frac{2}{n}I(\mathbf{W}_1;\mathbf{Y}_1)} \\ &\geq \sigma^2 (1 - \rho^2) \frac{N_1}{\epsilon + N_1} \quad \forall n \geq n_\epsilon, \end{aligned} \quad (22)$$

where the first inequality follows from rate-distortion theory (because  $W_{1,k}$  is  $\mathcal{N}(0, \sigma^2(1 - \rho^2))$ ), and the second inequality follows by Lemma A.1.

The third term in (19) is lower bounded in the following lemma.



**Lemma A.2.**

$$\frac{1}{n} \sum_{k=1}^n \mathbb{E}[(W_{1,k} - \mathbb{E}[W_{1,k}|\mathbf{Y}_1])(S_{2,k} - \mathbb{E}[S_{2,k}|\mathbf{Y}_1])] \geq -\sqrt{\frac{\epsilon}{N_1 + \epsilon}} \cdot \sigma. \quad (23)$$

*Proof.* See Appendix A.2.2.  $\square$

Combining the bounds in (20), (22) and (23) with the bound in (19) gives

$$\delta_1^{(n)} \geq \sigma^2(1 - \rho^2) \frac{N_1}{N_1 + \epsilon} - 2\rho \sqrt{\frac{\epsilon}{N_1 + \epsilon}} \cdot \sigma + \sigma^2 \rho^2 \frac{N_1}{N_1 + P}.$$

Taking the limit inferior as  $n \rightarrow \infty$  (with  $\epsilon > 0$  held fixed), and then letting  $\epsilon$  tend to zero, gives

$$\lim_{n \rightarrow \infty} \geq \sigma^2 \frac{N_1 + P(1 - \rho^2)}{N_1 + P},$$

and hence,

$$D_1^*(D_{2,\min}) \geq \sigma^2 \frac{N_1 + P(1 - \rho^2)}{N_1 + P}. \quad (24)$$

Since the RHS of (24) is achieved by the uncoded scheme with  $\alpha = 0$ ,  $\beta = 1$ , it follows that (24) must hold with equality, i.e., that

$$D_1^*(D_{2,\min}) = \sigma^2 \frac{N_1 + P(1 - \rho^2)}{N_1 + P}. \quad (25)$$

### A.2.1 Proof of Lemma A.1

The key element to the proof of Lemma A.1 is the following lemma.

**Lemma A.3.** *Any scheme resulting in the distortion  $\delta_2^{(n)}$  at Receiver 2, must produce a  $\mathbf{Y}_1$  satisfying*

$$I(\mathbf{S}_1; \mathbf{Y}_1 | \mathbf{S}_2) \leq \frac{n}{2} \log_2 \left( \frac{(P + N_2) \delta_2^{(n)} / \sigma^2 - N_2 + N_1}{N_1} \right). \quad (26)$$

*Proof.* We first notice that

$$\begin{aligned} I(\mathbf{S}_1; \mathbf{Y}_1 | \mathbf{S}_2) &= h(\mathbf{Y}_1 | \mathbf{S}_2) - h(\mathbf{Y}_1 | \mathbf{S}_1, \mathbf{S}_2) \\ &= h(\mathbf{Y}_1 | \mathbf{S}_2) - h(\mathbf{Z}_1) \\ &= h(\mathbf{Y}_1 | \mathbf{S}_2) - \frac{n}{2} \log_2 (2\pi e N_1). \end{aligned} \quad (27)$$

To upper bound  $I(\mathbf{S}_1; \mathbf{Y}_1 | \mathbf{S}_2)$  it thus suffices to upper bound  $h(\mathbf{Y}_1 | \mathbf{S}_2)$ . To this end, we first upper bound  $h(\mathbf{Y}_2 | \mathbf{S}_2)$  by means of rate-distortion theory, and then deduce an upper bound on  $h(\mathbf{Y}_1 | \mathbf{S}_2)$  by means of a conditional version of the entropy power inequality.

We denote the rate-distortion function for  $\mathbf{S}_2$  by  $R_{S_2}(\cdot)$  so that

$$R_{S_2}(\Delta_2) = \frac{1}{2} \log_2 \left( \frac{\sigma^2}{\Delta_2} \right),$$

for any  $\Delta_2 > 0$ . Hence,

$$\begin{aligned}
\frac{n}{2} \log_2 \left( \frac{\sigma^2}{\delta_2^{(n)}} \right) &= n R_{S_2}(\delta_2^{(n)}) \\
&\leq I(\mathbf{S}_2; \hat{\mathbf{S}}_2) \\
&\leq I(\mathbf{S}_2; \mathbf{Y}_2) \\
&= h(\mathbf{Y}_2) - h(\mathbf{Y}_2 | \mathbf{S}_2) \\
&\leq \frac{n}{2} \log_2 (2\pi e(P + N_2)) - h(\mathbf{Y}_2 | \mathbf{S}_2).
\end{aligned} \tag{28}$$

Rearranging (28) gives

$$\begin{aligned}
h(\mathbf{Y}_2 | \mathbf{S}_2) &\leq \frac{n}{2} \log_2 (2\pi e(P + N_2)) - \frac{n}{2} \log_2 \left( \frac{\sigma^2}{\delta_2^{(n)}} \right) \\
&= \frac{n}{2} \log_2 \left( 2\pi e(P + N_2) \frac{\delta_2^{(n)}}{\sigma^2} \right).
\end{aligned} \tag{29}$$

Based on (29) we now deduce an upper bound on  $h(\mathbf{Y}_1 | \mathbf{S}_2)$ . To this end, we first notice that for a sequence  $\{Z'_{2,k}\}$  that is IID  $\sim \mathcal{N}(0, N_1 - N_2)$  and independent of  $(\mathbf{Y}_1, \mathbf{S}_2)$ , we have that

$$h(\mathbf{Y}_2 | \mathbf{S}_2) = h(\mathbf{Y}_1 + \mathbf{Z}'_2 | \mathbf{S}_2).$$

Hence, by a conditional version of the entropy power inequality [8, Inequality (17)] it follows that

$$\begin{aligned}
2^{\frac{2}{n} h(\mathbf{Y}_2 | \mathbf{S}_2)} &= 2^{\frac{2}{n} h(\mathbf{Y}_1 + \mathbf{Z}'_2 | \mathbf{S}_2)} \\
&\geq 2^{\frac{2}{n} h(\mathbf{Y}_1 | \mathbf{S}_2)} + 2^{\frac{2}{n} h(\mathbf{Z}'_2)} \\
&= 2^{\frac{2}{n} h(\mathbf{Y}_1 | \mathbf{S}_2)} + 2\pi e(N_2 - N_1).
\end{aligned}$$

And thus,

$$\begin{aligned}
2^{\frac{2}{n} h(\mathbf{Y}_1 | \mathbf{S}_2)} &\leq 2^{\frac{2}{n} h(\mathbf{Y}_2 | \mathbf{S}_2)} - 2\pi e(N_2 - N_1) \\
&\leq 2\pi e(P + N_2) \frac{\delta_2^{(n)}}{\sigma^2} - 2\pi e(N_2 - N_1) \\
&= 2\pi e \left( (P + N_2) \frac{\delta_2^{(n)}}{\sigma^2} - N_2 + N_1 \right),
\end{aligned} \tag{30}$$

where in the second inequality we have used (29). Combining (30) with (27) gives

$$\begin{aligned}
I(\mathbf{S}_1; \mathbf{Y}_1 | \mathbf{S}_2) &\leq \frac{n}{2} \log_2 \left( 2\pi e \left( (P + N_2) \frac{\delta_2^{(n)}}{\sigma^2} - N_2 + N_1 \right) \right) - \frac{n}{2} \log_2 (2\pi e N_1) \\
&= \frac{n}{2} \log_2 \left( \frac{(P + N_2) \delta_2^{(n)} / \sigma^2 - N_2 + N_1}{N_1} \right).
\end{aligned} \quad \square$$

The proof of Lemma A.1 now follows easily.

*Proof of Lemma A.1.* The proof only requires applying Lemma A.3 to a sequence of schemes achieving  $D_{2,\min}$ . For such a sequence of schemes and for any  $\epsilon > 0$  there exists an integer  $n_\epsilon$  such that for all  $n \geq n_\epsilon$

$$\delta_2^{(n)} < D_{2,\min} + \frac{\epsilon \sigma^2}{N_2 + P} = \sigma^2 \frac{N_2 + \epsilon}{N_2 + P}. \quad (31)$$

By Lemma A.3

$$\left( \delta_2^{(n)} \leq \sigma^2 \frac{N_2 + \epsilon}{N_2 + P} \right) \Rightarrow \left( I(\mathbf{S}_1; \mathbf{Y}_1 | \mathbf{S}_2) \leq \frac{n}{2} \log_2 \left( \frac{\epsilon + N_1}{N_1} \right) \right).$$

And since  $I(\mathbf{S}_1; \mathbf{Y}_1 | \mathbf{S}_2) \geq I(\mathbf{S}_1 - \rho \mathbf{S}_2; \mathbf{Y}_1) = I(\mathbf{W}_1; \mathbf{Y}_1)$ , we obtain

$$\left( \delta_2^{(n)} \leq \sigma^2 \frac{N_2 + \epsilon}{N_2 + P} \right) \Rightarrow \left( I(\mathbf{W}_1; \mathbf{Y}_1) \leq \frac{n}{2} \log_2 \left( \frac{\epsilon + N_1}{N_1} \right) \right). \quad (32)$$

Combining (32) with (31) concludes the proof.  $\square$

### A.2.2 Proof of Lemma A.2

We first simplify the original expectation expression

$$\begin{aligned} & \frac{1}{n} \sum_{k=1}^n \mathbb{E}[(W_{1,k} - \mathbb{E}[W_{1,k} | \mathbf{Y}_1])(S_{2,k} - \mathbb{E}[S_{2,k} | \mathbf{Y}_1])] \\ & \stackrel{a)}{=} \frac{1}{n} \sum_{k=1}^n \mathbb{E}[(W_{1,k} - \mathbb{E}[W_{1,k} | \mathbf{Y}_1]) S_{2,k}] \\ & \stackrel{b)}{=} -\frac{1}{n} \sum_{k=1}^n \mathbb{E}[\mathbb{E}[W_{1,k} | \mathbf{Y}_1] S_{2,k}] \\ & \geq -\frac{1}{n} \sum_{k=1}^n \sqrt{\mathbb{E}[\mathbb{E}[W_{1,k} | \mathbf{Y}_1]^2]} \sqrt{\mathbb{E}[S_{2,k}^2]} \\ & \geq -\sqrt{\frac{1}{n} \sum_{k=1}^n \mathbb{E}[\mathbb{E}[W_{1,k} | \mathbf{Y}_1]^2]} \sqrt{\frac{1}{n} \sum_{k=1}^n \mathbb{E}[S_{2,k}^2]} \\ & = -\sqrt{\frac{1}{n} \sum_{k=1}^n \mathbb{E}[\mathbb{E}[W_{1,k} | \mathbf{Y}_1]^2]} \cdot \sigma, \end{aligned} \quad (33)$$

where *a)* follows since  $\mathbb{E}[S_{2,k} | \mathbf{Y}_1]$  is a function of  $\mathbf{Y}_1$  and hence is independent of  $(W_{1,k} - \mathbb{E}[W_{1,k} | \mathbf{Y}_1])$ , and *b)* follows since  $W_{1,k}$  is independent of  $S_{2,k}$ . The remaining square-root can now be bounded by means of (22):

$$\begin{aligned} \sigma^2(1 - \rho^2) \frac{N_1}{\epsilon + N_1} & \leq \frac{1}{n} \sum_{k=1}^n \mathbb{E}[(W_{1,k} - \mathbb{E}[W_{1,k} | \mathbf{Y}_1])^2] \\ & = \frac{1}{n} \sum_{k=1}^n \mathbb{E}[W_{1,k}^2] - 2 \frac{1}{n} \sum_{k=1}^n \mathbb{E}[W_{1,k} \mathbb{E}[W_{1,k} | \mathbf{Y}_1]] + \frac{1}{n} \sum_{k=1}^n \mathbb{E}[\mathbb{E}[W_{1,k} | \mathbf{Y}_1]^2] \\ & \stackrel{a)}{=} \frac{1}{n} \sum_{k=1}^n \mathbb{E}[W_{1,k}^2] - \frac{1}{n} \sum_{k=1}^n \mathbb{E}[\mathbb{E}[W_{1,k} | \mathbf{Y}_1]^2] \\ & = \sigma^2(1 - \rho^2) - \frac{1}{n} \sum_{k=1}^n \mathbb{E}[\mathbb{E}[W_{1,k} | \mathbf{Y}_1]^2], \end{aligned} \quad (34)$$

where  $a)$  follows since

$$\mathbb{E}\left[W_{1,k}\mathbb{E}[W_{1,k}|\mathbf{Y}_1]\right] = \mathbb{E}\left[\mathbb{E}[W_{1,k}|\mathbf{Y}_1]^2\right],$$

which holds by the orthogonality principle of the optimal reconstructor. Hence, rearranging (34) gives

$$\frac{1}{n} \sum_{k=1}^n \mathbb{E}\left[\mathbb{E}[W_{1,k}|\mathbf{Y}_1]^2\right] \leq \frac{\epsilon}{N_1 + \epsilon}.$$

Using this in (33), finally gives

$$\frac{1}{n} \sum_{k=1}^n \mathbb{E}[(W_{1,k} - \mathbb{E}[W_{1,k}|\mathbf{Y}_1])(S_{2,k} - \mathbb{E}[S_{2,k}|\mathbf{Y}_1])] \geq -\sqrt{\frac{\epsilon}{N_1 + \epsilon}} \cdot \sigma. \quad \square$$

## B Proof of Theorem 3.1

To prove Theorem 3.1 we need several preliminaries. Those are stated now.

**Remark B.1.** Theorem 3.1 is easily verified for  $(D'_1, D'_2) \in \mathcal{D}$  satisfying

$$D'_1 \geq \sigma^2 \frac{N_1 + P(1 - \rho^2)}{N_1 + P}. \quad (35)$$

To prove Theorem 3.1 for such pairs  $(D'_1, D'_2)$ , we simply show that for all  $P/N_1 \geq 0$ , every  $(D'_1, D'_2) \in \mathcal{D}$  satisfying (35) is achieved by the uncoded scheme. To see this, first note that by the definition of  $D_{2,\min}$ ,

$$D'_2 \geq D_{2,\min}, \quad (36)$$

whenever  $(D'_1, D'_2) \in \mathcal{D}$ . Also, by Proposition 2.1

$$D_1^*(D_{2,\min}) = \sigma^2 \frac{N_1 + P(1 - \rho^2)}{N_1 + P},$$

so, for  $(D'_1, D'_2) \in \mathcal{D}$  satisfying (35)

$$D'_1 \geq D_1^*(D_{2,\min}). \quad (37)$$

By Proposition 2.1 the pair  $(D_1^*(D_{2,\min}), D_{2,\min})$  is achieved by the uncoded scheme, and hence by (36) & (37) the same must be true for any pair  $(D'_1, D'_2) \in \mathcal{D}$  satisfying (35).

In view of Remark B.1 we shall assume in the rest of the proof that  $D_1$  satisfies

$$D_1 < \sigma^2 \frac{N_1 + P(1 - \rho^2)}{N_1 + P}. \quad (38)$$

Next, we define  $\tilde{D}_2^*(D_1)$  as the least distortion that can be achieved in estimating  $\mathbf{S}_2$  at Receiver 1 (!) subject to the constraint that Receiver 1 achieves a distortion  $D_1$  in estimating  $\mathbf{S}_1$ . More precisely:

**Definition B.1** ( $\tilde{D}_2^*(D_1)$ ). For every  $D_1 \geq D_{1,\min}$ , we define  $\tilde{D}_2^*(D_1)$  as

$$\tilde{D}_2^*(D_1) = \inf \{ \tilde{D}_2 \},$$

where the infimum is over all  $\tilde{D}_2$  to which there correspond average-power limited encoders  $\{f^{(n)}\}$  and reconstructors  $\{\phi_1^{(n)}\}, \{\tilde{\phi}_2^{(n)}\}$  satisfying

$$\begin{aligned} \overline{\lim}_{n \rightarrow \infty} \frac{1}{n} \sum_{k=1}^n \mathbb{E}[(S_{1,k} - \hat{S}_{1,k})^2] &\leq D_1, \\ \overline{\lim}_{n \rightarrow \infty} \frac{1}{n} \sum_{k=1}^n \mathbb{E}[(S_{2,k} - \tilde{S}_{2,k})^2] &\leq \tilde{D}_2, \end{aligned}$$

where  $\tilde{\phi}_2^{(n)}: \mathbf{Y}_1 \mapsto \tilde{\mathbf{S}}_2$  is any estimator of  $\mathbf{S}_2$  based on  $\mathbf{Y}_1$ , where  $\mathbf{X}$  is the result of applying  $f^{(n)}$  to  $(\mathbf{S}_1, \mathbf{S}_2)$ , and where  $\mathbf{Y}_1$  is the associated  $n$ -tuple received by Receiver 1.

**Remark B.2.** The distortion  $\tilde{D}_2^*(D_1)$  is the unique solution to the equation

$$R_{S_1, S_2}(D_1, \tilde{D}_2^*(D_1)) = \frac{1}{2} \log_2 \left( 1 + \frac{P}{N_1} \right), \quad (39)$$

where  $R_{S_1, S_2}(\cdot, \cdot)$  denotes the rate-distortion function on the pair  $S_1, S_2$  when it is observed by a common encoder, i.e.

$$R_{S_1, S_2}(\Delta_1, \Delta_2) = \min_{\substack{P_{T_1, T_2 | S_1, S_2}: \\ \mathbb{E}[(S_1 - T_1)^2] \leq \Delta_1 \\ \mathbb{E}[(S_2 - T_2)^2] \leq \Delta_2}} I(S_1, S_2; T_1, T_2).$$

The next proposition gives the explicit form of  $\tilde{D}_2^*(D_1)$  for the cases of interest to us.

**Proposition B.1.** Consider transmitting the bivariate Gaussian source (1) over the AWGN channel that connects the transmitter to Receiver 1. For any  $D_1$  satisfying (38) and  $P/N_1$  satisfying (13), the distortion  $\tilde{D}_2^*(D_1)$  is given by

$$\tilde{D}_2^*(D_1) = \sigma^2 \frac{P^2 \alpha^2 (1 - \rho^2) + P N_1 (\alpha^2 (2 - \rho^2) + 2 \alpha \beta \rho + \beta^2) + N_1^2 (\alpha^2 + 2 \alpha \beta \rho + \beta^2)}{(P + N_1)^2 (\alpha^2 + 2 \alpha \beta \rho + \beta^2)}, \quad (40)$$

where  $\alpha, \beta$  are such that  $D_1^u(\alpha, \beta) = D_1$ . Moreover, the pair  $(D_1, \tilde{D}_2^*(D_1))$  is achieved by the uncoded scheme with the above choice of  $\alpha$  and  $\beta$ .

*Proof.* For any  $D_1$  satisfying (38) and  $P/N_1$  satisfying (13), let  $\zeta$  denote the RHS of (40) with  $\alpha, \beta$  satisfying  $D_1^u(\alpha, \beta) = D_1$ . Using the explicit form of  $R_{S_1, S_2}(\cdot, \cdot)$ , as given in [1, Equation (10)], we obtain that  $R_{S_1, S_2}(D_1, \zeta)$  equals the RHS of (39). Thus, by Remark (B.2) it follows that  $\tilde{D}_2^*(D_1) = \zeta$ . Moreover, by (12) and our definition of  $\zeta$ , it follows that  $\zeta = D_2^u(\alpha, \beta)$  where  $\alpha, \beta$  are such that  $D_1^u(\alpha, \beta) = D_1$ . Thus,  $(D_1, \zeta)$ , i.e.,  $(D_1, \tilde{D}_2^*(D_1))$  is achieved by the uncoded scheme with that choice of  $\alpha, \beta$ .  $\square$

The heart of the proof of Theorem 3.1 is given in the following lemma. It characterizes the trade-off between the reconstruction fidelity  $D_1$  at Receiver 1 and the reconstruction fidelity  $D_2$  at Receiver 2.

**Lemma B.1.** *If the pair  $(D_1, D_2) \in \mathcal{D}$  satisfies (38), and if  $P/N_1$  satisfies (13), then for all real numbers  $a_1, a_2$  of equal sign,*

$$D_2 \geq \Psi(D_1, a_1, a_2), \quad (41)$$

where

$$\Psi(\delta, a_1, a_2) \triangleq \frac{\sigma^2}{P + N_2} \left( \frac{\sigma^2(1 - \rho^2)N_1}{\eta(\delta, a_1, a_2)} + N_2 - N_1 \right), \quad (42)$$

and where

$$\eta(\delta, a_1, a_2) = \sigma^2 - a_1(\sigma^2 - \delta)(2 - a_1) - a_2\sigma^2(2\rho - a_2) + 2a_1a_2\sqrt{(\sigma^2 - \delta)(\sigma^2 - \tilde{D}_2^*)},$$

where we have used the shorthand notation  $\tilde{D}_2^*$  for  $\tilde{D}_2^*(\delta)$ , which is given explicitly in Proposition B.1.

*Proof.* See Appendix B.1. □

We are now ready to prove Theorem 3.1.

*Proof of Theorem 3.1.* By Lemma B.1 it remains to verify that there exist real numbers  $a_1, a_2$  of equal sign such that  $(D_1, \Psi(D_1, a_1, a_2))$  coincides with the distortions achieved by the uncoded scheme. To this end, consider

$$a_1^* = \frac{(\sigma^2 - D_1)\sigma^2 - \rho\sigma^2\sqrt{(\sigma^2 - D_1)(\sigma^2 - \tilde{D}_2^*(D_1))}}{(\sigma^2 - D_1)\tilde{D}_2^*(D_1)}, \quad (43)$$

$$a_2^* = \frac{\rho\sigma^2 - \sqrt{(\sigma^2 - D_1)(\sigma^2 - \tilde{D}_2^*(D_1))}}{\tilde{D}_2^*(D_1)}. \quad (44)$$

We first show that  $a_1^*, a_2^*$  are both nonnegative, and thus indeed of equal sign. That  $a_1^*$  is nonnegative follows from (43) by noting that

$$D_1 < \sigma^2 \frac{N_1 + P(1 - \rho^2)}{N_1 + P} \quad \text{and} \quad \tilde{D}_2^*(D_1) \geq \sigma^2 \frac{N_1}{P + N_1},$$

where the upper bound on  $D_1$  is the one assumed in (38), and the lower bound on  $\tilde{D}_2^*$  follows by the classical single-user result [5, Theorem 9.6.3, p. 473]. To show that  $a_2^*$  is nonnegative, we distinguish between two cases. If  $D_1 \in (\sigma^2(1 - \rho^2), \sigma^2]$ , then the nonnegativity follows directly from (44) and from the fact that  $0 < D_2^*(D_1) \leq \sigma^2$ . Otherwise, if  $D_1 \in (0, \sigma^2(1 - \rho^2)]$ , then the nonnegativity of  $a_2^*$  follows from (44), using the inequality

$$D_2^*(D_1) \geq (\sigma^2(1 - \rho^2) - D_1) \frac{\sigma^2}{\sigma^2 - D_1},$$

an inequality which can be established using (39), the explicit form of  $R_{S_1, S_2}(\cdot, \cdot)$  [1, Equation (10)], and the assumption that  $P/N_1$  satisfies (13).

Having established that  $a_1^*$  and  $a_2^*$  are of equal sign, the proof now follows from Lemma B.1 by verifying that if  $(D_1, D_2) \in \mathcal{D}$  satisfies (38), and if  $P/N_1$  satisfies (13), then choosing  $(\alpha, \beta)$  so that  $D_1^u(\alpha, \beta) = D_1$  results in  $D_2^u(\alpha, \beta)$  satisfying

$$D_2^u(\alpha, \beta) = \Psi(D_1, a_1^*, a_2^*). \quad \square$$

## B.1 Proof of Lemma B.1

To prove Lemma B.1, we begin with a reduction.

**Reduction B.1.** To prove Lemma B.1 it suffices to consider pairs  $(D_1, D_2) \in \mathcal{D}$  that are achievable by coding schemes that achieve  $D_1$  with equality

$$\overline{\lim}_{n \rightarrow \infty} \frac{1}{n} \sum_{k=1}^n \mathbb{E} \left[ (S_{1,k} - \hat{S}_{1,k})^2 \right] = D_1, \quad (45)$$

and for which

$$\phi_i^{(n)}(\mathbf{Y}_i) = \mathbb{E}[\mathbf{S}_i | \mathbf{Y}_i] \quad i \in \{1, 2\}. \quad (46)$$

The proof of Reduction B.1 is based on the following lemma.

**Lemma B.2.** Any sequence of schemes achieving some boundary point  $(D_1, D_2^*(D_1))$  where  $D_1$  satisfies (38), must achieve both distortions with equality, i.e.

$$\overline{\lim}_{n \rightarrow \infty} \frac{1}{n} \sum_{k=1}^n \mathbb{E} \left[ (S_{1,k} - \hat{S}_{1,k})^2 \right] = D_1, \quad (47)$$

$$\overline{\lim}_{n \rightarrow \infty} \frac{1}{n} \sum_{k=1}^n \mathbb{E} \left[ (S_{2,k} - \hat{S}_{2,k})^2 \right] = D_2^*(D_1). \quad (48)$$

*Proof.* That  $D_2^*(D_1)$  must be achieved with equality by any sequence of schemes achieving  $(D_1, D_2^*(D_1))$ , follows from Definition 2.4 of  $D_2^*(D_1)$ .

We now show that if  $D_1$  satisfies (38), then also  $D_1$  must be achieved with equality. As we next show, to this end it suffices to show that for all  $D_1$  satisfying (38), the function  $D_2^*(\cdot)$  is strictly decreasing. Indeed, if  $D_2^*(\cdot)$  is strictly decreasing for all  $D_1$  satisfying (38), then a pair  $(D'_1, D_2^*(D_1))$  for any  $D_1$  satisfying (38) is achievable only if  $D'_1 \geq D_1$ . Hence, any sequence of schemes achieving  $(D_1, D_2^*(D_1))$  with  $D_1$  satisfying (38), must achieve  $D_1$  with equality.

It thus remains to show that for all  $D_1$  satisfying (38), the function  $D_2^*(\cdot)$ , which is illustrated in Figure 2, is strictly decreasing. By Proposition 2.1 we have that

$$D_1^*(D_{2,\min}) = \sigma^2 \frac{N_1 + P(1 - \rho^2)}{N_1 + P}. \quad (49)$$

From (49) it follows that

$$\left( D_1 < \sigma^2 \frac{N_1 + P(1 - \rho^2)}{N_1 + P} \right) \quad \Rightarrow \quad \left( D_2^*(D_1) > D_{2,\min} \right). \quad (50)$$

By the convexity of  $\mathcal{D}$  it follows that  $D_2^*(\cdot)$  is a convex function. This combines with (50) and our assumption that (38) holds, to imply that  $D_2^*(\cdot)$  is strictly decreasing in the interval<sup>3</sup>

$$\left( D_{1,\min}, \sigma^2 \frac{N_1 + P(1 - \rho^2)}{N_1 + P} \right],$$

where the interval's end point equals the RHS of (38).  $\square$

---

<sup>3</sup>Let  $g: (a, c) \rightarrow \mathbb{R}$  be a finite convex function and let  $b \in (a, c)$ . If  $b$  is such that

$$x < b \quad \Rightarrow \quad g(x) > g(b),$$

then  $g$  is strictly decreasing in the interval  $(a, b]$ . Here we apply this with  $a$  corresponding to  $D_{1,\min}$ , with  $b$  corresponding to the RHS of (38), and with  $c = \infty$ . This can be proved using [7, Corollary 24.2.1 and Theorem 24.1].

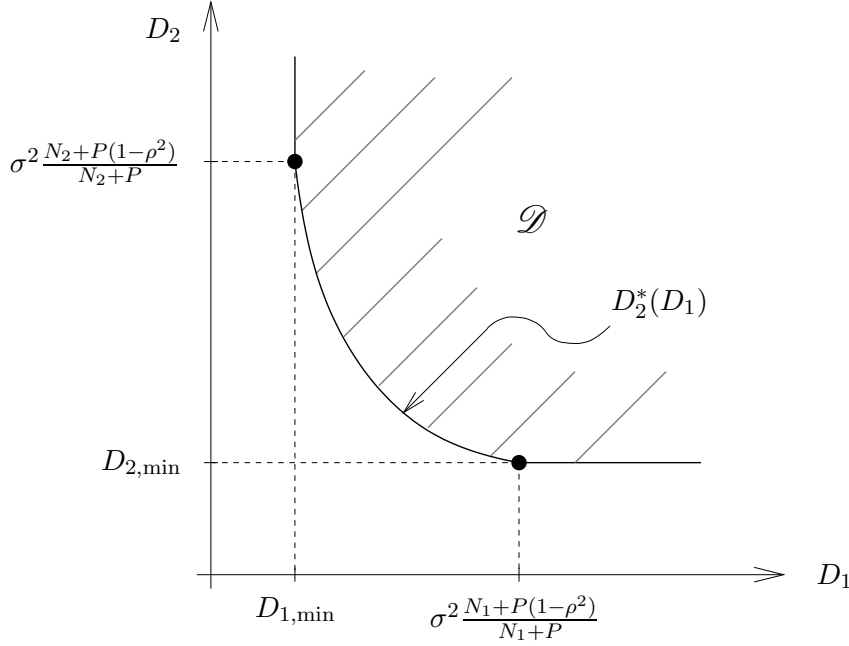


Figure 2: Monotonicity of  $D_2^*(\cdot)$ .

Based on Lemma B.2, the proof of Reduction B.1 follows easily.

*Proof of Reduction B.1.* The reduction to optimal reconstructors is straightforward. Since every  $(D_1, D_2) \in \mathcal{D}$  is achievable, it is certainly achievable by some sequence of schemes with optimal reconstructors.

It remains to prove that it suffices to limit ourselves to pairs  $(D_1, D_2) \in \mathcal{D}$  that are achievable by coding schemes that achieve  $D_1$  with equality. To this end, we first note that by Definition 2.4 it suffices to prove Lemma B.1 for pairs  $(D_1, D_2^*(D_1)) \in \mathcal{D}$  where  $D_1$  satisfies (13) and (38). The proof now follows by Lemma B.2 which states that for such pairs any sequence of schemes achieving  $(D_1, D_2^*(D_1))$  must achieve  $D_1$  with equality.  $\square$

To continue with the proof of Lemma B.1, we next derive a lower bound on  $\delta_2^{(n)}$  (for finite blocklengths  $n$ ).

**Lemma B.3.** *Let  $(f^{(n)}, \phi_1^{(n)}, \phi_2^{(n)})$  be a coding scheme where  $\phi_1^{(n)}$  and  $\phi_2^{(n)}$  satisfy (46). Then, for any  $a_1, a_2$  satisfying  $a_1 a_2 \geq 0$ ,*

$$\delta_2^{(n)} \geq \Psi(\delta_1^{(n)}, a_1, a_2). \quad (51)$$

Lemma B.3 relates the two reconstruction fidelities  $\delta_1^{(n)}$  and  $\delta_2^{(n)}$ . The difficulty in doing so is that if we consider a scheme achieving some  $\delta_2^{(n)}$  at Receiver 2, then we can only derive bounds on entropy expressions that are conditioned on  $\mathbf{S}_2$ . However, for a lower bound on  $\delta_1^{(n)}$  we would typically like to have an upper bound on  $I(\mathbf{S}_1; \hat{\mathbf{S}}_1)$ , or  $I(\mathbf{S}_1; \mathbf{Y}_1)$  (without conditioning on  $\mathbf{S}_2$ .) To overcome this difficulty, we furnish Receiver 1 with  $\mathbf{S}_2$  as side-information, and then prove Lemma B.3 using Lemma A.3 and the following upper bound.

**Lemma B.4.** *If a scheme  $(f^{(n)}, \phi_1^{(n)}, \phi_2^{(n)})$  satisfies the orthogonality condition*

$$\mathbb{E}[(S_{1,k} - \hat{S}_{1,k})\hat{S}_{1,k}] = 0 \quad \text{for every } 0 \leq k \leq n, \quad (52)$$



then

$$\frac{1}{n} \sum_{k=1}^n \mathbb{E} [\hat{S}_{1,k} S_{2,k}] \leq \sqrt{(\sigma^2 - \delta_1^{(n)}) (\sigma^2 - \tilde{D}_2^*(\delta_1^{(n)}))}. \quad (53)$$

*Proof.* The proof is based on the inequality

$$\frac{1}{n} \sum_{k=1}^n \mathbb{E} [(S_{2,k} - c \hat{S}_{1,k})^2] \geq \tilde{D}_2^*(\delta_1^{(n)}), \quad (54)$$

which holds for every  $c \in \mathbb{R}$  because the scaled sequence  $c\hat{\mathbf{S}}_1$  is a valid estimate of  $\mathbf{S}_2$  at Receiver 1. The desired bound now follows by evaluating the LHS of this inequality for the choice of

$$c = \sqrt{\frac{\sigma^2 - \tilde{D}_2^*}{\sigma^2 - \delta_1^{(n)}}}, \quad (55)$$

where we have used the shorthand notation  $\tilde{D}_2^*$  for  $\tilde{D}_2^*(\delta_1^{(n)})$ . Indeed, from (54) and (55) we obtain

$$\begin{aligned} \tilde{D}_2^* &\leq \frac{1}{n} \sum_{k=1}^n \mathbb{E} [(S_{2,k} - c \hat{S}_{1,k})^2] \\ &= \frac{1}{n} \sum_{k=1}^n \mathbb{E} [S_{2,k}^2] - 2c \frac{1}{n} \sum_{k=1}^n \mathbb{E} [S_{2,k} \hat{S}_{1,k}] + c^2 \frac{1}{n} \sum_{k=1}^n \mathbb{E} [\hat{S}_{1,k}^2] \\ &= \sigma^2 - 2 \sqrt{\frac{\sigma^2 - \tilde{D}_2^*}{\sigma^2 - \delta_1^{(n)}}} \frac{1}{n} \sum_{k=1}^n \mathbb{E} [S_{2,k} \hat{S}_{1,k}] + \sigma^2 - \tilde{D}_2^*, \end{aligned} \quad (56)$$

where in the last step we replaced  $c$  by its explicit value and used the property that the normalized summation over  $\mathbb{E} [\hat{S}_{1,k}^2]$  equals  $\sigma^2 - \delta_1^{(n)}$ , which follows from (52). Rearranging terms in (56) gives

$$\frac{1}{n} \sum_{k=1}^n \mathbb{E} [S_{2,k} \hat{S}_{1,k}] \leq \sqrt{(\sigma^2 - \delta_1^{(n)}) (\sigma^2 - \tilde{D}_2^*)}. \quad \square$$

We are now ready to prove Lemma B.3.

*Proof of Lemma B.3.* Denote by  $\Delta_1^{(n)}$  the least distortion that can be achieved on  $\mathbf{S}_1$  at Receiver 1 when  $\mathbf{S}_2$  is provided as side-information. The proof follows from a lower bound on  $\delta_2^{(n)}$  as a function of  $\Delta_1^{(n)}$  and from an upper bound on  $\Delta_1^{(n)}$  as a function of  $\delta_1^{(n)}$ .

We first derive the lower bound on  $\delta_2^{(n)}$ . To this end, let  $R_{S_1|S_2}(\cdot)$  denote the rate-distortion function on  $\mathbf{S}_1$  when  $\mathbf{S}_2$  is given as side-information to both, the encoder and the decoder. Thus, for every  $0 < \Delta_1 \leq \sigma^2(1 - \rho^2)$ ,

$$R_{S_1|S_2}(\Delta_1) = \frac{1}{2} \log_2 \left( \frac{\sigma^2(1 - \rho^2)}{\Delta_1} \right). \quad (57)$$

Since Receiver 1 is connected to the transmitter by a point-to-point link,

$$nR_{S_1|S_2}(\Delta_1^{(n)}) \leq I(\mathbf{S}_1; \mathbf{Y}_1 | \mathbf{S}_2). \quad (58)$$

The lower bound on  $\delta_2^{(n)}$  now follows from upper bounding the RHS of (58) by means of Lemma A.3, and rewriting the LHS of (58) using (57). This yields

$$\delta_2^{(n)} \geq \frac{\sigma^2}{P + N_2} \left( \frac{\sigma^2(1 - \rho^2)N_1}{\Delta_1^{(n)}} + N_2 - N_1 \right). \quad (59)$$

We next derive the upper bound on  $\Delta_1^{(n)}$  by considering the distortion of a linear estimator of  $\mathbf{S}_1$  when Receiver 1 has  $\mathbf{S}_2$  as side-information. More precisely, we consider the linear estimator

$$\check{S}_{1,k} = a_1 \hat{S}_{1,k} + a_2 S_{2,k}, \quad k \in \{1, \dots, n\},$$

where, as we will see, the coefficients  $a_1, a_2$  correspond to those in Lemma B.1. To analyze the distortion associated with  $\check{\mathbf{S}}_1$ , first note that by (46) the orthogonality condition of (52) is satisfied. Since  $\check{\mathbf{S}}_1$  is a valid estimate of  $\mathbf{S}_1$  at Receiver 1 when  $\mathbf{S}_2$  is given as side-information, we thus obtain

$$\begin{aligned} \Delta_1^{(n)} &\leq \frac{1}{n} \sum_{k=1}^n \mathbb{E}[(S_{1,k} - \check{S}_{1,k})^2] \\ &= \sigma^2 - 2a_1 \left( \frac{1}{n} \sum_{k=1}^n \mathbb{E}[S_{1,k} \hat{S}_{1,k}] \right) - 2a_2 \rho \sigma^2 + a_1^2 \left( \frac{1}{n} \sum_{k=1}^n \mathbb{E}[\hat{S}_{1,k}^2] \right) \\ &\quad + 2a_1 a_2 \left( \frac{1}{n} \sum_{k=1}^n \mathbb{E}[S_{1,k} \hat{S}_{2,k}] \right) + a_2^2 \sigma^2 \\ &\stackrel{a)}{=} \sigma^2 - 2a_1(\sigma^2 - \delta_1^{(n)}) - 2a_2 \rho \sigma^2 + a_1^2(\sigma^2 - \delta_1^{(n)}) \\ &\quad + 2a_1 a_2 \left( \frac{1}{n} \sum_{k=1}^n \mathbb{E}[S_{1,k} \hat{S}_{2,k}] \right) + a_2^2 \sigma^2, \\ &\stackrel{b)}{\leq} \sigma^2 - a_1(\sigma^2 - \delta_1^{(n)})(2 - a_1) - a_2 \sigma^2(2\rho - a_2) \\ &\quad + 2a_1 a_2 \sqrt{(\sigma^2 - \delta_1^{(n)}) (\sigma^2 - \tilde{D}_2^*(\delta_1^{(n)}))}. \end{aligned} \quad (60)$$

where in step *a*) we have used that the normalized summations over  $\mathbb{E}[\hat{S}_{1,k}^2]$  and  $\mathbb{E}[S_{1,k} \hat{S}_{2,k}]$  are both equal to  $\sigma^2 - \delta_1^{(n)}$ , which follows by (52); and in step *b*) we have used Lemma B.4 and the assumption that  $a_1 a_2 \geq 0$ .

The lower bound on  $\delta_2^{(n)}$  of Lemma B.3 now follows easily: Since the RHS of (59) is monotonically decreasing in  $\Delta_1^{(n)}$ , combining (60) with (59) gives

$$\delta_2^{(n)} \geq \frac{\sigma^2}{P + N_2} \left( \frac{\sigma^2(1 - \rho^2)N_1}{\eta(\delta_1^{(n)}, a_1, a_2)} + N_2 - N_1 \right),$$

where we have denoted by  $\eta(\delta_1^{(n)}, a_1, a_2)$  the RHS of (60).  $\square$

Based on Lemma B.3, the proof of Lemma B.1 now follows easily.

*Proof of Lemma B.1.* We show that for any nonnegative  $a_1, a_2$ , the achievable distortion  $D_2$  is lower bounded by

$$D_2 \geq \Psi(D_1, a_1, a_2).$$

By Reduction B.1 it suffices to show this for coding schemes  $\{f^{(n)}\}$ ,  $\{\phi_1^{(n)}\}$ ,  $\{\phi_2^{(n)}\}$  with  $\phi_1^{(n)}$  and  $\phi_2^{(n)}$  given in (46) and with associated normalized distortions  $\{\delta_1^{(n)}\}$ ,  $\{\delta_2^{(n)}\}$  satisfying

$$\overline{\lim}_{n \rightarrow \infty} \delta_1^{(n)} = D_1, \quad \text{and} \quad \overline{\lim}_{n \rightarrow \infty} \delta_2^{(n)} \leq D_2, \quad (61)$$

where  $D_1$  satisfies (38). By (61) there exists a subsequence  $\{n_k\}$ , tending to infinity, such that

$$\lim_{k \rightarrow \infty} \delta_1^{(n_k)} = D_1. \quad (62)$$

Hence,

$$\begin{aligned} D_2 &\stackrel{a)}{\geq} \overline{\lim}_{n \rightarrow \infty} \delta_2^{(n)} \\ &\geq \overline{\lim}_{k \rightarrow \infty} \delta_2^{(n_k)} \\ &\stackrel{b)}{\geq} \overline{\lim}_{k \rightarrow \infty} \Psi(\delta_1^{(n_k)}, a_1, a_2) \\ &\stackrel{c)}{=} \Psi(D_1, a_1, a_2), \end{aligned}$$

where  $a)$  follows from (61);  $b)$  follows from Lemma B.3; and  $c)$  follows from (62) and from the continuity of  $\Psi(\delta, a_1, a_2)$  with respect to  $\delta$  — a continuity which can be argued from (42) as follows. The function  $\Psi(\cdot)$  depends on  $\delta$  only through  $\eta(\delta, a_1, a_2)$ , and  $\eta(\delta, a_1, a_2)$  is strictly positive for all  $P/N_1 > 0$  and all  $a_1, a_2$ , and it is continuous in  $\delta$  because, by (40),  $\tilde{D}_2^*(\delta)$  is continuous in  $\delta$ . Hence,  $\Psi(\cdot)$  is continuous in  $\delta$ .  $\square$

## References

- [1] A. Lapidoth and S. Tinguely, “Sending a Bivariate Gaussian Source over a Gaussian MAC,” submitted to *IEEE Transactions on Information Theory*. Available on <http://arxiv.org/pdf/0901.3314>.
- [2] A. Lapidoth and S. Tinguely, “Sending a Bivariate Gaussian Source over a Gaussian MAC with Feedback,” submitted to *IEEE Transactions on Information Theory*. Available on <http://arxiv.org/pdf/0903.3487>.
- [3] T. J. Goblick, “Theoretical Limitations on the Transmission of Data from Analog Sources”, *IEEE Transaction on Information Theory*, IT-11(4): pp. 558-567, October 1965.
- [4] M. Gastpar, “Uncoded transmission is exactly optimal for a simple Gaussian sensor network”, in *Proceedings Information Theory and Applications Workshop*, San Diego, CA, USA, January 29 - February 2, 2007.
- [5] R. G. Gallager, *Information Theory and Reliable Communication*, John Wiley & Sons, 1968.
- [6] T. M. Cover and J. A. Thomas, *Elements of Information Theory*, New York, John Wiley & Sons, 1991.
- [7] R. T. Rockafellar, *Convex Analysis*, Princeton Univeristy Press, New Jersey, 1970.
- [8] N. Blachmann, “The Convolution Inequality for Entropy Powers”, *IEEE Transactions on Information Theory*, vol. IT-11(2), pp. 267-271, April 1965.